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SOME GEOMETRIC INVESTIGATIONS ON THE GENERAL PROBLEM OF DYNAMICS.

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Received January 20, 1920. Presented February 11, 1920.

§1. **Physical Interpertations.** 1) *Kinetics.* Consider the motion of a system in a conservative field of force from one position P_0 to another P_1, with the sum of its kinetic and potential energies equal to a given constant; the system will move naturally or unguided along the path for which the *action* is least. This is known as Jacobi's *principle of least action.* Action is defined as twice the time-integral of the kinetic energy T, i. e.

$$Action = \int_{(P_0)}^{(P_1)} 2\,T\,dt.$$

If U is the potential energy, W is the work function (negative potential), h is the given constant of energy, v is the velocity of the system, then, for a unit mass, we have the energy equation

$$T + U = h, \qquad \text{or} \qquad \tfrac{1}{2}v^2 - W = h, \qquad \text{or} \qquad v^2 = 2(W + h),$$

and

$$\int 2T\,dt = \int v^2\,dt = \int v \cdot v\,dt = \int v\,ds = \int \sqrt{2\,(W + h)}\,ds.$$

The totality of trajectories for all initial conditions are thus determined according to the principle of least action by

$$Action = \int_{(P_0)}^{(P_1)} v\,ds = \int_{(P_0)}^{(P_1)} \sqrt{2\,(W + h)}\,ds = minimum.$$

2) *Brachistochrones.* If the conservative system described above moves not according to the principle of least action but so that the time of motion from P_0 to P_1 is least, the path is called a *brachistochrone,* and the totality of trajectories for all initial conditions are thus determined by

$$Time = \int_{(P_0)}^{(P_1)} dt = \int_{(P_0)}^{(P_1)} \frac{ds}{v} = \int_{(P_0)}^{(P_1)} \frac{ds}{\sqrt{2\,(W + h)}} = minimum.$$

3) *Optics.* The paths of light in an isotropic medium in which the index of refraction ν varies from point to point, are determined according to Fermat's principle by

$$\int_{(P_0)}^{(P_1)} \nu \, ds = minimum.$$

Here the integral is proportional to the time.

4) *Catenaries.* When a homogeneous, flexible, inextensible string is acted on by conservative forces, the forms of equilibrium (general catenaries) are determined by

$$\int_{(P_0)}^{(P_1)} 2\,(W + h)\, ds = minimum.$$

5) *General case*; *natural families.* We note that each one of the above problems leads to a set of curves determined by an expression of the form

$$\int_{(P_0)}^{(P_1)} F \, ds = minimum.$$

where F is a function of the coördinates of position only, and ds is the element of length in the space under consideration. The curves defined by such an expression may be called, following Painlevé, a natural family. For the dynamical trajectories, brachistochrones, and catenaries, there are ∞^1 such families corresponding to all possible values of the constant h; but in the optical case there is only one such family.

§2. **The general problem of dynamics.** Consider a system with n degrees of freedom, i. e. the position of the system at any instant is determined by n independent parameters or coördinates; thus

$$x = f_1(x_1, x_2, \ldots, x_n),\quad y = f_2(x_1, x_2, \ldots, x_n),\quad z = f_3(x_1, x_2, \ldots, x_n).$$

Then

$$\frac{dx}{dt} = \frac{\partial f_1}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f_1}{\partial x_n}\frac{dx_n}{dt},\quad \frac{dy}{dt} = \frac{\partial f_2}{\partial x_1}\frac{dx_1}{dt} + \ldots + \frac{\partial f_2}{\partial x_n}\frac{dx_n}{dt},\quad \text{etc.,}$$

and twice the kinetic energy is

$$2T = v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \sum_{ik} a_{ik}\frac{dx_i}{dt}\frac{dx_k}{dt} \quad (i, k = 1, 2, \ldots, n),$$

where the a's are functions of the n parameters $x_1, x_2, \ldots, x_n$. If the acting forces are conservative, there will exist a work function $W(x_1, x_2, \ldots, x_n)$ independent of the time, and a constant of energy h so that $T - W = h$. The trajectories are then determined by

$$\int_{(P_0)}^{(P_1)} \sqrt{2(W+h)} \quad \sqrt{\sum_k a_{ik}\, dx_i\, dx_k} = minimum.$$

Employing geometric language, we may regard $x_1, x_2, \ldots, x_n$ as the coördinates of a point in a space of n dimensions, and P_0 and P_1 as two points in such a space. As the element of arc length in such a space is given by

$$ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,$$

we are led to the problem of minimizing $\int F\, ds$, where F is a function of the coördinates $x_1, x_2, \ldots x_n$ and ds is the element of arc in a space of n dimensions. The space may be euclidean (of zero curvature), or non-euclidean, of constant or variable curvature. The natural family consists of $\infty^{2(n-1)}$ curves, one through each point in each direction.

If the work function W is zero, then F is a constant, and our problem reduces to finding the curves for which

$$\int_{(P_0)}^{(P_1)} ds = minimum;$$

this evidently defines the *geodesics* in our space.

§3. **Geometric Properties of a dynamical system.** In 1869, Beltrami [1] proved a remarkable theorem concerning the geodesics in a space of n dimensions, V_n. This theorem may be stated as follows:

The ∞^{n-1} *geodesics which pass out normally from points of any hypersurface,* V_{n-1}, *form a normal hypercongruence, i. e. admits* ∞^1 *normal hypersurfaces, which are the loci of equal arc lengths measured from any one of the system of hypersurfaces.*

The theorem is a generalization of Gauss's theorem for the geodesics on an ordinary surface, V_2.

In 1871, Lipschitz [2] announced the corresponding theorem for the general problem of dynamics; using the geometric language of natural families, this theorem may be stated as follows:

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[1] E. Beltrami, *Sulla teorica generale dei parametri differenziali.* Memorie dell' Academia delle Scienze dell' Instituto di Bologna, series 2a, vol. 8, p. 549.

[2] R. Lipschitz, *Untersuchung eines Problems der Variationsrechnung inwelchem das Problem der Mechanik enthalten ist.* Crelle's Journal, vol. 74.

The ∞^{n-1} *curves of a natural family which pass out normally to any hypersurface,* V_{n-1}, *form a normal hypercongruence, i. e. admit of* ∞^1 *normal hypersurfaces, which are the loci of equal values of* $\int F\,ds$ *measured from any one of the system of hypersurfaces.*

This theorem is true for space of any dimensionality or any curvature.[3]

Thomson and Tait[4] had given this theorem for dynamical trajectories in a conservative field of force in a euclidean space of three dimensions. We quote: "If from all points on an arbitrary surface, particles not mutually influencing one another be projected with the proper velocities [so as to make the sum of the kinetic and potential energies have a given value] in the direction of the normals; points which they reach with equal actions lie on a surface cutting the paths at right angles." The ∞^1 orthogonal surfaces appear as surfaces of equal action. A similar theorem may be stated using the language of brachistochrones, or of optical light paths; in these cases the ∞^1 orthogonal surfaces appear as surfaces of equal time.

The question arises whether the orthogonal properties stated above are *characteristic* of the trajectories under the principle of least action, or the brachistochrones, or the optical light waves, or general catenaries, or, in short, of *any* natural family? In other words, does the orthogonal property belong exclusively to natural families of curves? This question may be answered in the affirmative. Using geometric language we may state the theorem:

If a system of $\infty^{2(n-1)}$ *curves in space of n dimensions (euclidean or curved) is such that* ∞^{n-1} *curves of the system meet an arbitrary hypersurface (space of* $n-1$ *dimensions) orthogonally, and always form a normal hypercongruence, i. e. admit of* ∞^1 *orthogonal hypersurfaces, the system is of the natural type.*

Thus the system may be considered as any one of the types, dynamical or optical, discussed above. This *converse* of the Lipschitz theorem or of a generalized Thomson and Tait theorem was first proved for three dimensional spaces of constant curvature by Kasner.[5] In the following sections the author proves this theorem for a euclidean space of four dimensions (§9), then for a euclidean space of n dimensions, (§10), and finally for a general curved space of n dimensions, (§11). §5 contains a derivation of the differential equations of the natural family of curves by the methods of the calculus of variation. In §6 the author gives a very simple and direct proof of the Lipschitz theorem. It will be noted that although the proof of the direct theorem is very brief, the proof of the converse theorem is quite long.

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[3] For a detailed discussion of these theorems and of the general problem of dynamics, see G. Darboux, Leçons sur la théorie générale des surfaces, vol. 2, chapt. 8. See also, P. Appell, Traité de mécanique rationelle, vol. 2, chapt. 25.

[4] Thomson and Tait, *Treatise on Natural Philosophy,* vol. 1, part 1, p. 353, 1879.

The methods adopted for proving the converse theorems are similar to those used by Kasner for the three dimensional case, except that the use of the arc length as the parameter along a trajectory introduces a symmetry without which the rather long and complex algebraic expressions could hardly be handled.[6]

§4. **Distance and angle.** The totality of points determined by assigning all possible values to n variables or coördinates $x_1, x_2, \ldots, x_n$ are said to constitute a space of n dimensions, V_n. If the x's are expressed as functions of m variables, $u_1, u_2, \ldots, u_m$,

$$x_i = f_i(u_1, u_2, \ldots, u_m), \quad (i = 1, 2, \ldots, n)$$

the totality of points thus determined are said to form a space of m dimensions, V_m, contained in V_n. In particular, if $m = n - 1$, we have a hypersurface of $n - 1$ dimensions; if $m = 1$, we have a curve.

The element of arc length or distance between two points x_i and $x_i + dx_i$ in V_n is defined by a quadratic differential form

$$(1) \qquad ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,^{7}$$

where the a's are functions of the x's. The parameter curve x_i is such that along it only the coördinate x_i varies, the other coördinates remaining constant; along such a curve the element of arc length evidently takes the form

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[5] E. Kasner, *The theorem of Thomson and Tait and natural families of trajectories*. Trans. Am. Math. Soc., vol. 11 (1910), pp. 121–140.

[6] Natural families of curves have been characterized geometrically in other ways, for euclidean spaces by E. Kasner: *Natural families of trajectories: conservative fields of force;* Trans. Am. Math. Soc., vol. 10 (1909), pp. 201–219; and for general curved spaces by the present writer: *Natural families of curves in a general curved space of n-dimensions;* Trans. Am. Math. Soc., vol. 13 (1912), pp. 77–95.

[7] We write only $\sum_{ik}$ and understand that the summation is to be carried out from 1 to n for each of the indicated subscripts, unless otherwise specified.

$$ds_i = \sqrt{a_{ii}}\, dx_i. \tag{2}$$

A direction passing out from a point x_i is determined by the n direction cosines

$$\xi_i = \frac{dx_i}{ds}, \quad (i = 1, 2, \ldots, n) \tag{3}$$

which are related by the identity

$$\sum_{ik} a_{ik}\, \xi_i\, \xi_k = 1. \tag{4}$$

If $\xi_i^{(1)}$ and $\xi_i^{(2)}$ determine two directions passing out from a point x_i, the angle between them is given by

$$\cos \omega = \sum_{ik} a_{ik}\, \xi_i^{(1)}\, \xi_k^{(2)}, \tag{5}$$

and hence the angle between two parameter curves x_i and x_k is

$$\cos \omega_{ik} = \frac{a_{ik}}{\sqrt{a_{ii}\, a_{kk}}}. \tag{6}$$

If the space is euclidean, the parameter lines at any point may be chosen as n mutually orthogonal straight lines, hence, from (6) and (2),

$$a_{ik} = 0 \quad \text{if} \quad i \neq k, \quad \text{and} \quad a_{ii} = 1 \quad (i, k = 1, 2, \ldots, n) \tag{7}$$

and the element of arc length takes the form

$$ds^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2. \tag{8}$$

§5. **Differential equations of a natural family of curves.** Our problem is to find the differential equations of the curves for which

$$t = \int_{(P_0)}^{(P_1)} F\, ds = \int_{t_0}^{t_1} F \sqrt{\sum_{ik} a_{ik}\, dx_i\, dx_k} = \textit{minimum}. \tag{9}$$

The variation of this integral, keeping the end points P_0 and P_1 fixed must then be equal to zero. Using the usual symbol δ as the symbol of variation, we have

$$dt^2 = F^2\, ds^2 = \sum_{ik} (F^2 a_{ik})\, dx_i\, dx_k,$$

$$2\, dt\, \delta\,(dt) = \sum_{ikl} \frac{\partial\, (F^2 a_{ik})}{\partial x_l}\, dx_i\, dx_k\, \delta x_l + 2 \sum_{il} (F^2 a_{il})\, dx_i\, \delta\,(dx_l)$$

$$\therefore \delta\,(dt) = \tfrac{1}{2} \sum_{ikl} \frac{\partial\, (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt}\, dt\, \delta x_l + \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, \delta\,(dx_l):$$

Integrating both sides, and noting that, by partial integration,

$$\int_{t_0}^{t_1} \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, d\,(\delta x_l) = \left| \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, \delta x_l \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt} \sum_{il} (F^2 a_{il}) \frac{dx_i}{dt}\, dt\, \delta x_l$$

and that the first term of the right member vanishes since δx_l is zero at the fixed end points P_0 and P_1, we finally get

$$\delta \int_{t_0}^{t_1} dt = \int_{t_0}^{t_1} \sum_{l} \left[\sum_{ik} \tfrac{1}{2} \frac{\partial\, (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt} - \frac{d}{dt} \sum_{i} (F^2 a_{il}) \frac{dx_i}{dt} \right] dt\, \delta x_l.$$

Since δx_l is arbitrary, this expression will vanish if

$$\frac{d}{dt} \sum_{i} F^2 a_{il} \frac{dx_i}{dt} = \tfrac{1}{2} \sum_{ik} \frac{\partial (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{dt} \frac{dx_k}{dt}, \quad (l = 1, 2, \ldots., n).$$

Replacing dt by $F\, ds$, this becomes

$$(10) \quad \frac{d}{ds} \sum_{i} F a_{il} \frac{dx_i}{ds} = \frac{1}{2F} \sum_{ik} \frac{\partial\, (F^2 a_{ik})}{\partial x_l} \frac{dx_i}{ds} \frac{dx_k}{ds}, \quad (l = 1, 2, \ldots, n).$$

For a *euclidean* space, we may use the relations (7), and (10) becomes

$$(11) \quad \frac{d}{ds}\left(F \frac{dx_l}{ds} \right) = \frac{1}{2F} \sum_{i} \frac{\partial F^2}{\partial x_l} \left(\frac{dx_i}{ds} \right)^2, \quad (l = 1, 2, \ldots., n)$$

or, expanding and solving for $\frac{d^2 x_l}{ds^2}$,

$$\frac{d^2 x_l}{ds^2} = \frac{1}{F} \left[\frac{\partial F}{\partial x_l} \sum_{i} \left(\frac{dx_i}{ds} \right)^2 - \frac{dx_l}{ds} \sum_{k} \frac{\partial F}{\partial x_k} \frac{dx_k}{ds} \right], \quad (l = 1, 2, \ldots., n).$$

If we let $L = \log F$, write[8] $x'_l = \frac{dx_l}{ds}$, $x''_l = \frac{d^2 x_l}{ds^2}$, and choose the arc length s as the parameter along the curves, so that

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[8] Throughout this paper, primes refer to total derivatives with respect to s.

$$(12) \qquad \sum_i (x'_i)^2 = 1,$$

we get

$$(13) \qquad x''_l = \frac{\partial L}{\partial x_l} - x'_l \sum_k \frac{\partial L}{\partial x_k} x'_k, \quad (l = 1, 2, \ldots\ldots, n),$$

as the differential equations of the natural family in euclidean space.

For *any* space, expanding (10), we get

$$\sum_i a_{il}\, x''_i + \sum_{ik} \left(\frac{\partial a_{il}}{\partial x_k} - \tfrac{1}{2} \frac{\partial a_{ik}}{\partial x_l}\right) x'_i\, x'_k = \frac{1}{F}\left[\frac{\partial F}{\partial x_l} \sum_{ik} a_{ik}\, x'_i\, x'_k - \sum_{ik} \frac{\partial F}{\partial x_k}\, a_{il}\, x'_i\, x'_k\right].$$

Choosing the arc length as parameter along the curves, so that

$$(4) \qquad \sum_{ik} a_{ik}\, x'_i\, x'_k = 1,$$

writing $L = \log F$, and introducing the Christoffel three-indices symbol of the first kind [9]

$$\begin{bmatrix} ik \\ l \end{bmatrix} = \tfrac{1}{2}\left(\frac{\partial a_{il}}{\partial x_k} + \frac{\partial a_{kl}}{\partial x_i} - \frac{\partial a_{ik}}{\partial x_l}\right),$$

this becomes

$$\sum_i a_{il}\, x''_i + \sum_{ik} \begin{bmatrix} ik \\ l \end{bmatrix} x'_i\, x'_k = \frac{\partial L}{\partial x_l} - \sum_{ik} \frac{\partial L}{\partial x_k}\, a_{il}\, x'_i\, x'_k \qquad (l = 1, 2, \ldots\ldots, n).$$

To solve these equations for x'', we multiply by A_{ml}, the minor of a_{ml} in the determinant $a = |\, a_{\lambda\mu} \,|$ divided by a itself, and sum with respect to l. From the properties of the determinant $|\, a_{\lambda\mu} \,|$ we have

$$(14) \qquad \sum_l a_{il}\, A_{ml} = 0, \quad \text{if } i \pm m; \qquad \sum_l a_{il}\, A_{il} = 1.$$

Our equations then become

$$(15) \qquad x''_m + \sum_{ik} \begin{Bmatrix} ik \\ m \end{Bmatrix} x'_i\, x'_k = \sum_l \frac{\partial L}{\partial x_l} (A_{ml} - x'_m\, x\, l), \quad (m = 1, 2, \ldots\ldots, n),$$

where we have introduced the Christoffel symbol of the second kind

$$\begin{Bmatrix} ik \\ m \end{Bmatrix} = \sum_l A_{ml} \begin{bmatrix} ik \\ l \end{bmatrix}.$$

Equations (15) reduce immediately to equations (13) if we place

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[9] Bianchi, *Geometria differenziale*, vol. 1, chapt. II.

$$(16) \qquad A_{ml} = 0, \text{ if } l \neq m; \qquad A_{mm} = 1, \qquad (l, m = 1, 2, \ldots, n).$$

Thus *any natural family of curves is represented by differential equations of the form* (13) *for euclidean space and of the form* (15) *for any space. Here L is an arbitrary point function.* From these equations it is evident that, given a point x_i of V_n and a direction ξ_i through this point, one and only one curve of a given natural family of curves is determined. There are thus ∞^{n-1} curves passing through a point and a totality of $\infty^{2(n-1)}$ curves composing the family.

If the function F is constant, our problem reduces to finding the curves for which

$$(17) \qquad \int ds = \textit{minimum},$$

and equations (15) become

$$(18) \qquad x''_m + \sum_{ik} \left\{ {ik \atop m} \right\} x'_i x'_k = 0, \quad (m = 1, 2, \ldots, n)$$

the well-known equations of the *geodesics* in any space.[10]

§6. **Proof of the direct or Lipschitz theorem.** Darboux[11] proves this by noting that the determination of the *geodesics* in any space V_n, where

$$ds^2 = \sum_{ik} F^2 a_{ik}\, dx_i\, dx_k,$$

leads to a minimizing of

$$\int \sqrt{\sum_{ik} F^2 a_{ik}\, dx_i\, dx_k} = \int F \sqrt{\sum_{ik} a_{ik}\, dx_i\, dx_k}\,.$$

If, now, the *geodesics* in the space V_n are *conformally* represented in a space V'_n in which

$$ds^2 = \sum_{ik} a_{ik}\, dx_i\, dx_k,$$

the representing curves will evidently form a *natural* family in V'_n. Since conformal representation leaves angles unchanged, the orthogonal property expressed by Beltrami's theorem for the geodesics will be immediately carried over into the orthogonal property expressed by Lipschitz's theorem for a natural family. Further, since ds is transformed into $F\,ds$, the hypersurfaces which are the loci of equal arc lengths are transformed into the hypersurfaces which are the loci of equal actions.

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[10] Bianchi, ibid., vol. 1, p. 334.
[11] Darboux, ibid., vol. 2, p. 510.

We shall now give a more direct proof of the Lipschitz theorem without using the Beltrami Theorem as a basis.

Consider the equations of the natural family given by

$$(10)\quad \frac{d}{ds}\sum_i Fa_{il}\frac{dx_i}{ds} = \frac{1}{2F}\sum_{ik}\frac{\partial(F^2a_{ik})}{\partial x_l}\frac{dx_i}{ds}\frac{dx_k}{ds}, \quad (l = 1, 2, \ldots, n).$$

Let the parameter curves x_1 be curves of the natural type, and let the parameter x_1 represent the action along these curves measured from the corresponding intersection with a definite orthogonal parameter hypersurface of the system $x_1 =$ constant, e. g. $x_1 = 0$. Thus

$$x_1 = \int F_1\,ds_1 = \int F_1\sqrt{a_{11}}\,dx_1,$$

since, along the x_1 parameter curves, we have by (2)

$$ds_1 = \sqrt{a_{11}}\,dx_1,$$

so that

$$F_1\sqrt{a_{11}} = 1.$$

Applying equations (10) to our x_1 parameter curves (along which $x_2, x_3, \ldots, x_n$ are all constants), we have

$$(19)\qquad \frac{\partial}{\partial x_1}\left(\frac{a_{1l}}{a_{11}}\right) = 0, \quad (l = 2, 3, \ldots, n).$$

Since the parameter curves x_1 are normal to the hypersurface $x_1 = 0$, we have from (6) for the angle between the parameter curve x_1 and any other parameter curve x_l,

$$\cos\omega_{1l} = \frac{a_{1l}}{\sqrt{a_{11}}\sqrt{a_{ll}}} = 0, \text{ for } x_1 = 0$$

or

$$\frac{a_{1l}}{a_{11}} = 0, \quad (l = 2, 3, \ldots, n), \text{ for } x_1 = 0.$$

But as shown by (19), a_{1l}/a_{11} is independent of x_1, so that

$$\frac{a_{1l}}{a_{11}} = 0, \quad (l = 2, 3, \ldots, n), \text{ for all values of } x_1$$

and hence the curves x_1 are normal to all hypersurfaces $x_1 =$ constant, and they thus admit of ∞^1 normal hypersurfaces, i. e. they form a normal hypercongruence; the hypersurfaces appear as surfaces of equal action.

§7. **Conditions for a normal hypercongruence.** Consider first a *euclidean space of four dimensions*, S_4. We wish to derive the conditions that a system of ∞^3 curves in S_4 are such that there exists a system of ∞^1 hypersurfaces (V_3) which are orthogonal to the curves. We write the equations of the ∞^3 curves in the form

$$(20)\quad X = X\,(u, v, w, t),\quad Y = Y\,(u, v, w, t),\quad Z = Z\,(u, v, w, t),\quad R = R\,(u, v, w, t)$$

where u, v, w are three arbitrary parameters whose particular values u_0, v_0, w_0 determine uniquely a curve C_0, while the variable t determines the points on this curve.

We shall now assume that there is a hypersurface H which is orthogonal to the curves; let this be determined by placing

$$(21)\qquad t = t\,(u, v, w)$$

in equations (20). Through a point (X, Y, Z, R) of such a hypersurface, determined by the values $u = u_0, v = v_0, w = w_0$, there passes the curve

$$(22)\quad X = X\,(u_0, v_0, w_0, t),\quad Y = Y\,(u_0\, v_0, w_0, t),\quad Z = Z\,(u_0, v_0, w_0, t),\quad R = R\,(u_0, v_0, w_0, t)$$

and the direction-cosines of its tangent are proportional to

$$\frac{\partial X}{dt}, \quad \frac{\partial Y}{\partial t}, \quad \frac{\partial Z}{\partial t}, \quad \frac{\partial R}{\partial t}.$$

Since this direction is perpendicular to our surface, we must have

$$(23)\qquad \frac{\partial X}{\partial t}\,dX + \frac{\partial Y}{\partial t}\,dY + \frac{\partial Z}{\partial t}\,dZ + \frac{\partial R}{\partial t}\,dR = 0,$$

where dX, dY, dZ, dR are computed from (20) and t is replaced by its value from (21). Now

$$dX = \frac{\partial X}{\partial u}\,du + \frac{\partial X}{\partial v}\,dv + \frac{\partial X}{\partial w}\,dw + \frac{\partial X}{\partial t}\,dt,$$

and similarly for dY, dZ, and dR. Introducing these values in (23), we get

$$T\,dt + U\,du + V\,dv + W\,dw = 0 \tag{24}$$

where

$$T = \Sigma\left(\frac{\partial X}{\partial t}\right)^2,\quad U = \Sigma\frac{\partial X}{\partial t}\frac{\partial X}{\partial u},\quad V = \Sigma\frac{\partial X}{\partial t}\frac{\partial X}{\partial v},\quad W = \Sigma\frac{\partial X}{\partial t}\frac{\partial X}{\partial w}, \tag{25}$$

the summation extending over X, Y, Z, and R. The value of t must therefore be a function of u, v, w which satisfies equation (24). In order that we may have a system of hypersurfaces which are orthogonal to these curves, it is necessary and sufficient that the conditions for integrability of equation (24) be satisfied for all values of t, u, v, w. These conditions are

$$\begin{aligned}
&T\left(\frac{\partial U}{\partial v}-\frac{\partial V}{\partial u}\right)+U\left(\frac{\partial V}{\partial t}-\frac{\partial T}{\partial v}\right)+V\left(\frac{\partial T}{\partial u}-\frac{\partial U}{\partial t}\right)=0\\
&T\left(\frac{\partial V}{\partial w}-\frac{\partial W}{\partial v}\right)+V\left(\frac{\partial W}{\partial t}-\frac{\partial T}{\partial w}\right)+W\left(\frac{\partial T}{\partial v}-\frac{\partial V}{\partial t}\right)=0\\
&T\left(\frac{\partial W}{\partial u}-\frac{\partial U}{\partial w}\right)+W\left(\frac{\partial U}{\partial t}-\frac{\partial T}{\partial u}\right)+U\left(\frac{\partial T}{\partial w}-\frac{\partial W}{\partial t}\right)=0,
\end{aligned} \tag{26}$$

where the second and third of these conditions may be derived from the first by a cyclical interchange of U, V, W and of u, v, w.

For a *euclidean space of n dimensions*, S_n, the equations of the ∞^{n-1} curves may be written

$$X_i = X_i\,(u_1, u_2, \ldots, u_{n-1}, t), \quad (i = 1, 2, \ldots, n). \tag{20'}$$

The orthogonal hypersurface H is determined by placing

$$t = t\,(u_1, u_2, \ldots, u_{n-1}) \tag{21'}$$

in (20′), and through a point $X_i^{(0)}$ of such a hypersurface, there passes the curve

$$X_i = X_i\,(u_1^{(0)}, u_2^{(0)}, \ldots, u_{n-1}^{(0)}, t), \quad (i = 1, 2, \ldots n), \tag{22'}$$

and the direction-cosines of its tangent are proportional to

$$\frac{\partial X_i}{\partial t}, \quad (i = 1, 2, \ldots, n).$$

Since this direction is perpendicular to our surface, we must have

$$\sum_i \frac{\partial X_i}{\partial t} dX_i = 0. \tag{23'}$$

where dX_i is computed from (20′) and t is replaced by its value from (21′). Now

$$dX_i = \sum_{r=1}^{n-1} \frac{\partial X_i}{\partial u_r} du_r + \frac{\partial X_i}{\partial t} dt, \quad (i = 1, 2, \ldots, n).$$

Introducing these values in (23′), we get

$$T\,dt + U_1\,du_1 + U_2\,du_2 + \ldots + U_{n-1}\,du_{n-1} = 0, \tag{24'}$$

where

$$T = \sum_i \left(\frac{\partial X_i}{\partial t}\right)^2, \quad U_r = \sum_i \frac{\partial X_i}{\partial t}\frac{\partial X_i}{\partial u_r}, \quad (r = 1, 2, \ldots, n-1). \tag{25'}$$

The conditions of integrability of equation (24′) are

$$T\left(\frac{\partial U_k}{\partial u_r} - \frac{\partial U_r}{\partial u_k}\right) + U_k\left(\frac{\partial U_r}{\partial t} - \frac{\partial T}{\partial u_r}\right) + U_r\left(\frac{\partial T}{\partial u_k} - \frac{\partial U_k}{\partial t}\right) = 0, \tag{26'}$$

$$(r, k = 1, 2, \ldots, n-1).$$

Equations (26′) *give* $\frac{(n-1)(n-2)}{2}$ *independent conditions; these must be satisfied if our system of curves is to form a normal hypercongruence.*

Equations (26′) are also the conditions for a normal hypercongruence in *any space of n dimensions*, V_n, if we note that the condition (23′) for perpendicularity is replaced by

$$\sum_{ik} a_{ik} \frac{\partial X_i}{\partial t} dX_k = 0, \tag{23''}$$

so that equations (25′) are replaced by

$$T = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial t}, \quad U_r = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial u_r}, \quad (r=1,2,\ldots,n-1). \tag{25''}$$

§8. **Analytic statement of the converse theorem.** Consider an arbitrary system of $\infty^{2(n-1)}$ curves in a euclidean space of n dimensions, assuming that one curve passes through each point in each direction. Such a system may be defined by a set of n differential equations of the second order—

$$(27)\quad x_i'' = F_i\,(x_1, x_2, \ldots, x_n;\quad x_1', x_2', \ldots x_n'),\quad (i = 1, 2, \ldots, n),$$

where $x_i' = \dfrac{dx_i}{ds}, x_i'' = \dfrac{d^2x_i}{ds^2}$, and the arc length s is chosen as parameter along the curves, so that

$$(28)\qquad \sum_i (x_i')^2 = 1.$$

Here the F_i are uniform functions, analytic in the 2n arguments. *Our problem is to find the form of the function* F_i *so that the* ∞^{n-1} *curves of the system which pass out orthogonally to any hypersurface in the space, should form a normal hypercongruence.*

Denoting the initial values of x_i, x_i', which may be taken at random, subject to (28), by x_i, p_i respectively, and employing $X_1, X_2, \ldots, X_n$ as current coördinates, we may write the solution of (27) in the form

$$(29)\quad X_i = x_i + p_i\,(S - s) + \tfrac{1}{2}\,F_i\,(S - s)^2 + \tfrac{1}{6}\,M_i\,(S - s)^3 + \ldots,\quad (i = 1, 2, \ldots, n),$$

$$(30)\qquad \sum_i p_i^2 = 1.$$

Here the F_i are expressed as functions of x_i, p_i, and the M_i found by differentiating (27) are given by

$$(31)\qquad M_i = \sum_l \left(\frac{\partial F_i}{\partial x_l}p_l + \frac{\partial F_i}{\partial p_l}F_l\right),\quad (i = 1, 2, \ldots, n).$$

Consider, now, an arbitrary hypersurface

$$(32)\qquad x_i = f_i\,(u_1, u_2, \ldots, u_{n-1}),\quad (i = 1, 2, \ldots, n).$$

At each point of the surface and normal to it a *definite* curve of the given system (27) may be constructed. A certain hypercongruence will thus be determined. We wish to express the conditions that this shall be of the normal type. If the direction of the normal at any point of the surface is given by $P_i\,(u_1, u_2, \ldots, u_{n-1})$, $(i = 1, 2, \ldots, n)$, then these P_i are determined by the equations

$$(33)\qquad \sum_i P_i^2 = 1;\quad \sum_i P_i\frac{\partial f_i}{\partial u_r} = 0,\quad (r = 1, 2, \ldots, n - 1).$$

The second of these conditions merely expresses the fact that the normal is perpendicular to the parameter curves of the surface (32). By differentiating (33) we have the further relations

$$(34)\quad \sum_i P_i \frac{\partial P_i}{\partial u_r} = 0,\ (r = 1, 2, \ldots, n-1);\ \sum_i \frac{\partial P_i}{\partial u_k}\frac{\partial f_i}{\partial u_r} = \sum_i \frac{\partial P_i}{\partial u_r}\frac{\partial f_i}{\partial u_k},\quad (r, k = 1, 2, \ldots, n-1).$$

We may now identify the normal at any point of the surface with the tangent line to the curve, so that

$$(35)\qquad p_i = P_i\,(u_1, u_2, \ldots, u_{n-1}),\quad (i = 1, 2, \ldots, n).$$

The equations of the ∞^{n-1} curves corresponding to the given initial conditions may now be written

$$(36)\quad X_i = f_i + P_i t + \tfrac{1}{2}\overline{F}_i t^2 + \tfrac{1}{6}\overline{M}_i t^3 + \ldots,\quad (i = 1, 2, \ldots, n),$$

where t replaces $S - s$, and where, e. g.,

$$(37)\quad \overline{F}_i\,(u_1, u_2, \ldots, u_{n-1}) = F_i\,(f_i, f_2, \ldots, f_n;\ \ P_1,\ P_2, \ldots, P_n).$$

The coefficients of the powers of t in (36) are thus functions of $(n-1)$ parameters $u_1, u_2, \ldots, u_{n-1}$, so that the equations of the ∞^{n-1} curves are expressed in the form (20′). In order that the system (36) form a normal hypercongruence, equations (26′) must be satisfied.

Applying (25′) and (26′) to (36), and using the conditions (33) and (34), we get

$$\frac{\partial X_i}{\partial t} = P_i + \overline{F}_i t + \ldots;\quad \frac{\partial X_i}{\partial u_r} = \frac{\partial f_i}{\partial u_r} + t\frac{\partial P_i}{\partial u_r} + \ldots;\quad \frac{\partial X_i}{\partial u_k} = \frac{\partial f_i}{\partial u_k} + t\frac{\partial P_i}{\partial u_k} + \ldots.$$

$$T = \sum_i P_i^2 + 2t\sum_i P_i F_i + \ldots = 1 + 2t\sum_i P_i F_i + \ldots,$$

$$U_r = \sum_i P_i \frac{\partial f_i}{\partial u_r} + t\left(\sum_i P_i \frac{\partial P_i}{\partial u_r} + \sum_i \overline{F}_i \frac{\partial f_i}{\partial u_r}\right) + \ldots = t\sum_i \overline{F}_i \frac{\partial f_i}{\partial u_r} + \ldots,$$

$$U_k = \sum_i P_i \frac{\partial f_i}{\partial u_k} + t\left(\sum_i P_i \frac{\partial P_i}{\partial u_k} + \sum_i F_i \frac{\partial f_i}{\partial u_k}\right) + \ldots = t\sum_i F_i \frac{\partial f_i}{\partial u_k} + \ldots$$

$$\frac{\partial T}{\partial u_r} = 2t\left(\sum_i P_i \frac{\partial f_i}{\partial u_r} + \sum_i \overline{F}_i \frac{\partial P_i}{\partial u_r}\right) + \ldots;\quad \frac{\partial T}{\partial u_k} = 2t\left(\sum_i P_i \frac{\partial f_i}{\partial u_k} + \sum_i \overline{F}_i \frac{\partial P_i}{\partial u_k}\right) + \ldots$$

$$\frac{\partial U_r}{\partial t} = \sum_i F_i \frac{\partial f_i}{\partial u_r} + \ldots; \quad \frac{\partial U_r}{\partial u_k} = t\left(\sum_i \overline{F}_i \frac{\partial^2 f_i}{\partial u_r \partial u_k} + \sum_i \frac{\partial \overline{F}_i}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) + \ldots.$$

$$\frac{\partial U_k}{dt} = \sum_i \overline{F}_i \frac{\partial f_i}{\partial u_k} + \ldots; \quad \frac{\partial U_k}{\partial u_r} = t\left(\sum_i \overline{F}_i \frac{\partial^2 f_i}{\partial u_k \partial u_r} + \sum_i \frac{\partial \overline{F}_i}{\partial u_r}\frac{\partial f_i}{\partial u_k}\right) + \ldots.$$

Substituting in (26′) we shall get our condition in the form

$$\alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \ldots\ldots\ldots = 0. \tag{38}$$

Since this must vanish independent of t, every coefficient, α, must vanish. We find that α_0 vanishes identically, which is as it should be since the hypercongruence is normal to the initial surface $t = 0$ by hypothesis. Let us first discuss the condition that α_1 vanish. This gives

$$\sum_i \frac{\partial \overline{F}_i}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \sum_i \frac{\partial \overline{F}_i}{\partial u_k}\frac{\partial f_i}{\partial u_r} = 0, \quad (r, k = 1, 2, \ldots, n-1). \tag{39}$$

Now by differentiating (37), we get

$$\frac{\partial \overline{F}_i}{\partial u_r} = \sum_l \frac{\partial F_i}{\partial x_l}\frac{\partial f_l}{\partial u_r} + \sum_l \frac{\partial F_i}{\partial p_l}\frac{\partial P_l}{\partial u_r}$$

$$\frac{\partial \overline{F}_i}{\partial u_k} = \sum_l \frac{\partial F_i}{\partial x_l}\frac{\partial f_l}{\partial u_k} + \sum_l \frac{\partial F_i}{\partial p_l}\frac{\partial P_l}{\partial u_k} \tag{40}$$

and (39) becomes

$$\sum_{il} \frac{\partial F_i}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_i}{\partial u_r} - \frac{\partial f_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) + \sum_{il} \frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) = 0, \tag{41}$$

$$(r, k = 1, 2, \ldots, n-1).$$

These $\frac{(n-1)(n-2)}{2}$ *conditions are therefore necessary conditions in order that the* ∞^{n-1} *curves belonging to the system* (27) *and orthogonal to the hypersurface* (32) *shall form a normal hypercongruence.*

Our problem is to find the forms of the functions F_i in order that this condition of orthogonality should hold for *every* hypersurface (32) taken as a base. Equations (41) must then necessarily hold for n arbitrary functions f_i $(i = 1, 2, \ldots, n)$. These functions may be so chosen that for any assigned values of $u_1, u_2, \ldots, u_{n-1}$, the quantities

f_i and all partial derivatives of all orders of f_i with respect to the u's, shall take on *arbitrary* numerical values, or so that the quantities f_i, P_i, $\frac{\partial f_i}{\partial u_r}$, $\frac{\partial P_i}{\partial u_r}$, shall take on arbitrary numerical values, subject only to the relations (33) and (34). Before applying these conditions in a euclidean space of n dimensions, it will be of advantage to apply them to a euclidean space of four dimensions.

§9. **Proof of the converse theorem in a euclidean space of 4 dimensions.** For a space of four dimensions we may write the equations of the base hypersurface (32) in the form

$$(32') \qquad x_i = f_i(u, v, w), \quad (i = 1, 2, 3, 4),$$

and the six conditions (33) and (34) as

$$(33') \qquad P_1 \frac{\partial f_1}{\partial u} + P_2 \frac{\partial f_2}{\partial u} + P_3 \frac{\partial f_3}{\partial u} + P_4 \frac{\partial f_4}{\partial u} = 0, \quad (u \to v \to w)^{12}$$

$$(34') \qquad \left(\frac{\partial P_1}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right) + \left(\frac{\partial P_3}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_3}{\partial v}\right)$$
$$+ \left(\frac{\partial P_4}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial P_4}{\partial u}\frac{\partial f_4}{\partial v}\right) = 0, \quad (u \to v \to w)^{12}$$

The three necessary conditions (41) become

$$(41') \qquad \sum_{il}^{1\ldots4} \frac{\partial F_i}{\partial x_l}\left(\frac{\partial f_l}{\partial v}\frac{\partial f_i}{\partial u} - \frac{\partial f_l}{\partial u}\frac{\partial f_i}{\partial v}\right) + \sum_{il}^{1\ldots4} \frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial v}\frac{\partial f_i}{\partial u} - \frac{\partial P_l}{\partial u}\frac{\partial f_i}{\partial v}\right) = 0,$$
$$(u \to v \to w).$$

We may evidently place each of the two summations in the left member of this equation equal to zero.

Expanding the first summation in (41′) we may evidently group the terms into the following eighteen:

$$(42) \left(\frac{\partial F_2}{\partial x_3} - \frac{\partial F_3}{\partial x_2}\right)\left(\frac{\partial f_3}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial f_3}{\partial u}\frac{\partial f_2}{\partial v}\right) + \left(\frac{\partial F_3}{\partial x_1} - \frac{\partial F_1}{\partial x_3}\right)\left(\frac{\partial f_1}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial f_1}{\partial u}\frac{\partial f_3}{\partial v}\right)$$
$$+ \left(\frac{\partial F_1}{\partial x_2} - \frac{\partial F_2}{\partial x_1}\right)\left(\frac{\partial f_2}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial f_2}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left(\frac{\partial F_4}{\partial x_1} - \frac{\partial F_1}{\partial x_4}\right)\left(\frac{\partial f_1}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial f_1}{\partial u}\frac{\partial f_4}{\partial v}\right)$$

---

[12] We shall use the symbol $(u \to v \to w)$ to mean that similar relations may be written by a cyclical interchange of u, v, w.

$$+\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)\left(\frac{\partial f_2}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial f_2}{\partial u}\frac{\partial f_4}{\partial v}\right)+\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)\left(\frac{\partial f_3}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial f_3}{\partial u}\frac{\partial f_4}{\partial v}\right)$$
$$= 0, \; (u \to v \to w)$$

To eliminate the f's we proceed as follows. Solve (33′) for the ratios of the p's, thus

(43) $p_1 : p_2 : p_3 : p_4 =$

$$-\begin{vmatrix} \frac{\partial f_4}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_4}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \\ \frac{\partial f_4}{\partial w} & \frac{\partial f_2}{\partial w} & \frac{\partial f_3}{\partial w} \end{vmatrix} : -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_4}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_4}{\partial v} & \frac{\partial f_3}{\partial v} \\ \frac{\partial f_1}{\partial w} & \frac{\partial f_4}{\partial w} & \frac{\partial f_3}{\partial w} \end{vmatrix} : -\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_4}{\partial u} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_4}{\partial v} \\ \frac{\partial f_1}{\partial w} & \frac{\partial f_2}{\partial w} & \frac{\partial f_4}{\partial w} \end{vmatrix} : \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} \\ \frac{\partial f_1}{\partial v} & \frac{\partial f_2}{\partial v} & \frac{\partial f_3}{\partial v} \\ \frac{\partial f_1}{\partial w} & \frac{\partial f_2}{\partial w} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

Multiply the three equations of (42) by $\frac{\partial f_1}{\partial w}, \frac{\partial f_1}{\partial u}, \frac{\partial f_1}{\partial v}$, respectively, and add, remembering the expansions of the determinants (43) for the p's in terms of minors of the second order. We get

$$p_4\left(\frac{\partial F_2}{\partial x_3}-\frac{\partial F_3}{\partial x_2}\right)+p_3\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)-p_2\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)=0.$$

Similarly, multiply the three equations (42) by $\frac{\partial f_2}{\partial w}, \frac{\partial f_2}{\partial u}, \frac{\partial f_2}{\partial v}$, respectively, by $\frac{\partial f_3}{\partial w}, \frac{\partial f_3}{\partial u}, \frac{\partial f_3}{\partial v}$, respectively, and by $\frac{\partial f_4}{\partial w}, \frac{\partial f_4}{\partial u}, \frac{\partial f_4}{\partial v}$, respectively, and add. We thus get the four conditions

$$(44)\quad \begin{cases} p_4\left(\frac{\partial F_2}{\partial x_3}-\frac{\partial F_3}{\partial x_2}\right)+p_3\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)-p_2\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)=0 \\ p_4\left(\frac{\partial F_3}{\partial x_1}-\frac{\partial F_1}{\partial x_3}\right)-p_3\left(\frac{\partial F_4}{\partial x_1}-\frac{\partial F_1}{\partial x_4}\right)+p_1\left(\frac{\partial F_4}{\partial x_3}-\frac{\partial F_3}{\partial x_4}\right)=0 \\ p_4\left(\frac{\partial F_1}{\partial x_2}-\frac{\partial F_2}{\partial x_1}\right)+p_2\left(\frac{\partial F_4}{\partial x_1}-\frac{\partial F_1}{\partial x_4}\right)-p_1\left(\frac{\partial F_4}{\partial x_2}-\frac{\partial F_2}{\partial x_4}\right)=0 \\ p_1\left(\frac{\partial F_2}{\partial x_3}-\frac{\partial F_3}{\partial x_2}\right)+p_2\left(\frac{\partial F_3}{\partial x_1}-\frac{\partial F_1}{\partial x_3}\right)+p_3\left(\frac{\partial F_1}{\partial x_2}-\frac{\partial F_2}{\partial x_1}\right)=0 \end{cases}$$

Now, expanding the second summation in (41′), we get the following 48 terms:

$$\begin{aligned}
&\frac{\partial F_1}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_1}{\partial v}\right)+\frac{\partial F_1}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_1}{\partial v}\right)\\
&+\frac{\partial F_1}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_1}{\partial v}\right)+\frac{\partial F_1}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_1}{\partial v}\right)\\
&+\frac{\partial F_2}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right)+\frac{\partial F_2}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_2}{\partial v}\right)\\
&+\frac{\partial F_2}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_2}{\partial v}\right)+\frac{\partial F_2}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_2}{\partial v}\right)\\
&+\frac{\partial F_3}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_3}{\partial v}\right)+\frac{\partial F_3}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_3}{\partial v}\right)\\
&+\frac{\partial F_3}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_3}{\partial v}\right)+\frac{\partial F_3}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_3}{\partial v}\right)\\
&+\frac{\partial F_4}{\partial p_4}\left(\frac{\partial P_4}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_4}{\partial v}\right)+\frac{\partial F_4}{\partial p_1}\left(\frac{\partial P_1}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_1}{\partial u}\frac{\partial f_4}{\partial v}\right)\\
&+\frac{\partial F_4}{\partial p_2}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_4}{\partial v}\right)+\frac{\partial F_4}{\partial p_3}\left(\frac{\partial P_3}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_3}{\partial u}\frac{\partial f_4}{\partial v}\right),\\
&\qquad\qquad\qquad\qquad\qquad\qquad (u\rightarrow v\rightarrow w).
\end{aligned}$$

We may reduce the number of terms by eliminating the multipliers of $\frac{\partial F_1}{\partial p_2}, \frac{\partial F_2}{\partial p_3}, \frac{\partial F_3}{\partial p_4}, \frac{\partial F_4}{\partial p_1}$. Thus to eliminate the coefficient of $\frac{\partial F_1}{\partial p_2}$, we solve (33′) for

$$\begin{aligned}
\frac{\partial f_1}{\partial u}&=-\frac{p_2}{p_1}\frac{\partial f_2}{\partial u}-\frac{p_3}{p_1}\frac{\partial f_3}{\partial u}-\frac{p_4}{p_1}\frac{\partial f_4}{\partial u},\\
\frac{\partial f_1}{\partial v}&=-\frac{p_2}{p_1}\frac{\partial f_2}{\partial v}-\frac{p_3}{p_1}\frac{\partial f_3}{\partial v}-\frac{p_4}{p_1}\frac{\partial f_4}{\partial v},
\end{aligned}$$

and thus get

$$\begin{aligned}
\frac{\partial P_2}{\partial v}\frac{\partial f_1}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_1}{\partial v}=-\frac{p_2}{p_1}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right)&-\frac{p_3}{p_1}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_3}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_3}{\partial v}\right)\\
&-\frac{p_4}{p_1}\left(\frac{\partial P_2}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_2}{\partial u}\frac{\partial f_4}{\partial v}\right).
\end{aligned}$$

If we further eliminate

$$\frac{\partial P_4}{\partial v}\frac{\partial f_4}{\partial u}-\frac{\partial P_4}{\partial u}\frac{\partial f_4}{\partial v}$$

by means of (34′), the above 48 terms reduce to 33, viz.:

$$
\begin{aligned}
(45)\quad & \left[\frac{\partial F_1}{\partial p_3} - \frac{p_1}{p_2}\frac{\partial F_2}{\partial p_3}\right]\left(\frac{\partial P_3}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left[\frac{\partial F_2}{\partial p_4} - \frac{p_2}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_4}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_4}{\partial u}\frac{\partial f_2}{\partial v}\right) \\
& + \left[\frac{\partial F_3}{\partial p_1} - \frac{p_3}{p_4}\frac{\partial F_4}{\partial p_1}\right]\left(\frac{\partial P_1}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_3}{\partial v}\right) + \left[\frac{\partial F_4}{\partial p_2} - \frac{p_4}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_4}{\partial v}\right) \\
& + \left[\frac{\partial F_1}{\partial p_4} - \frac{p_1}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_4}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_4}{\partial u}\frac{\partial f_1}{\partial v}\right) + \left[\frac{\partial F_2}{\partial p_1} - \frac{p_2}{p_4}\frac{\partial F_4}{\partial p_1}\right]\left(\frac{\partial P_1}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_2}{\partial v}\right) \\
& + \left[\frac{\partial F_3}{\partial p_2} - \frac{p_3}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_3}{\partial v}\right) + \left[\frac{\partial F_4'}{\partial p_3} - \frac{p_4}{p_2}\frac{\partial F_2}{\partial p_3}\right]\left(\frac{\partial P_3}{\partial v}\frac{\partial f_4}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_4}{\partial v}\right) \\
& + \left[\frac{\partial F_1}{\partial p_1} - \frac{p_1}{p_4}\frac{\partial F_4}{\partial p_1} - \frac{\partial F_4}{\partial p_4} + \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_1}{\partial v}\frac{\partial f_1}{\partial u} - \frac{\partial P_1}{\partial u}\frac{\partial f_1}{\partial v}\right) \\
& + \left[\frac{\partial F_2}{\partial p_2} - \frac{p_2}{p_1}\frac{\partial F_1}{\partial p_2} - \frac{\partial F_4}{\partial p_4} + \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_2}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial P_2}{\partial u}\frac{\partial f_2}{\partial v}\right) \\
& + \left[\frac{\partial F_3}{\partial f_3} - \frac{p_3}{p_2}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_4}{\partial p_4} + \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}\right]\left(\frac{\partial P_3}{\partial v}\frac{\partial f_3}{\partial u} - \frac{\partial P_3}{\partial u}\frac{\partial f_3}{\partial v}\right) = 0, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (u \to v \to w).
\end{aligned}
$$

The coefficients in parentheses in (45) may now be taken as independent quantities, and for (45) to vanish identically, we must have the vanishing of all the expressions in brackets. Hence, the first 8 terms give

$$
\begin{aligned}
& p_2\frac{\partial F_1}{\partial p_3} - p_1\frac{\partial F_2}{\partial p_3} = 0; \quad p_3\frac{\partial F_2}{\partial p_4} - p_2\frac{\partial F_3}{\partial p_4} = 0; \quad p_4\frac{\partial F_3}{\partial p_1} - p_3\frac{\partial F_4}{\partial p_1} = 0; \\
& p_1\frac{\partial F_4}{\partial p_2} - p_4\frac{\partial F_1}{\partial p_2} = 0; \quad p_3\frac{\partial F_1}{\partial p_4} - p_1\frac{\partial F_3}{\partial p_4} = 0; \quad p_4\frac{\partial F_2}{\partial p_1} - p_2\frac{\partial F_4}{\partial p_1} = 0; \\
& p_1\frac{\partial F_3}{\partial p_2} - p_3\frac{\partial F_1}{\partial p_2} = 0; \quad p_2\frac{\partial F_4}{\partial p_3} - p_4\frac{\partial F_2}{\partial p_3} = 0.
\end{aligned}
$$

By combinations of these we get four additional relations

$$p_1\frac{\partial F_4}{\partial p_3} - p_4\frac{\partial F_1}{\partial p_3} = 0;\ p_2\frac{\partial F_1}{\partial p_4} - p_1\frac{\partial F_2}{\partial p_4} = 0;\ p_3\frac{\partial F_2}{\partial p_1} - p_2\frac{\partial F_3}{\partial p_1} = 0;$$

$$p_4\frac{\partial F_3}{\partial p_2} - p_3\frac{\partial F_4}{\partial p_2} = 0.$$

These 12 relations may be written in the equivalent forms

$$\frac{\partial}{\partial p_3}\left(p_2F_1 - p_1F_2\right) = 0;\ \frac{\partial}{\partial p_4}\left(p_2F_1 - p_1F_2\right) = 0;\ \frac{\partial}{\partial p_1}\left(p_3F_2 - p_2F_3\right) = 0;$$

$$\frac{\partial}{\partial p_4}\left(p_3F_2 - p_2F_3\right) = 0;\ \frac{\partial}{\partial p_1}\left(p_4F_3 - p_3F_4\right) = 0;\ \frac{\partial}{\partial p_2}\left(p_4F_3 - p_3F_4\right) = 0;$$

$$\frac{\partial}{\partial p_2}\left(p_1F_4 - p_4F_1\right) = 0;\ \frac{\partial}{\partial p_3}\left(p_1F_4 - p_4F_1\right) = 0;\ \frac{\partial}{\partial p_2}\left(p_3F_1 - p_1F_3\right) = 0;$$

$$\frac{\partial}{\partial p_4}\left(p_3F_1 - p_1F_3\right) = 0;\ \frac{\partial}{\partial p_1}\left(p_4F_2 - p_2F_4\right) = 0;\ \frac{\partial}{\partial p_3}\left(p_4F_2 - p_2F_4\right) = 0.$$

Hence

$$(46)\quad \begin{cases} p_2F_1 - p_1F_2 = \alpha_{12};\ p_3F_2 - p_2F_3 = \alpha_{23};\ p_4F_3 - p_3F_4 = \alpha_{34}; \\ p_1F_4 - p_4F_1 = \alpha_{41};\ p_3F_1 - p_1F_3 = \alpha_{13};\ p_4F_2 - p_2F_4 = \alpha_{24}; \end{cases}$$

where, e. g.,

$$(47)\quad \alpha_{12} = \alpha_{12}(p_1, p_2, x_1, x_2, x_3, x_4);\ \alpha_{23} = \alpha_{23}(p_2, p_3, x_1, x_2, x_3, x_4);\ \text{etc.}$$

The vanishing of the last three brackets in (45) gives

$$(48)\quad \frac{\partial F_1}{\partial p_1} - \frac{p_1}{p_4}\frac{\partial F_4}{\partial p_1} = \frac{\partial F_2}{\partial p_2} - \frac{p_2}{p_1}\frac{\partial F_1}{\partial p_2} = \frac{\partial F_3}{\partial p_3} - \frac{p_3}{p_2}\frac{\partial F_2}{\partial p_3} = \frac{\partial F_4}{\partial p_4} - \frac{p_4}{p_3}\frac{\partial F_3}{\partial p_4}.$$

These are conditions on the forms of the α's appearing in (46). To find these conditions we may proceed as follows: in

$$\frac{\partial F_1}{\partial p_1} - \frac{p_1}{p_4}\frac{\partial F_4}{\partial p_1} = \frac{\partial F_3}{\partial p_3} - \frac{p_3}{p_2}\frac{\partial F_2}{\partial p_3}$$

we may replace $\frac{\partial F_4}{\partial p_1}$ by $\frac{p_4}{p_2}\frac{\partial F_2}{\partial p_1}$, and get

$$p_2\frac{\partial F_1}{\partial p_1} - p_1\frac{\partial F_2}{\partial p_1} = p_2\frac{\partial F_3}{\partial p_3} - p_3\frac{\partial F_2}{\partial p_3}$$

Adding F_2 to both members of this equation, we evidently have

$$\frac{\partial}{dp_1}(p_2F_1 - p_1F_2) = \frac{\partial}{\partial p_3}(p_2F_3 - p_3F_2)$$

or

$$\frac{\partial \alpha_{12}}{\partial p_1} + \frac{\partial \alpha_{23}}{\partial p_3} = 0.$$

By similar procedure we finally get the conditions

$$(49)\quad \left\{\begin{array}{lll} \dfrac{\partial \alpha_{12}}{\partial p_1} + \dfrac{\partial \alpha_{23}}{\partial p_3} = 0; & \dfrac{\partial \alpha_{23}}{\partial p_2} + \dfrac{\partial \alpha_{34}}{\partial p_4} = 0; & \dfrac{\partial \alpha_{34}}{\partial p_3} + \dfrac{\partial \alpha_{41}}{\partial p_1} = 0; \\ \dfrac{\partial \alpha_{41}}{\partial p_4} + \dfrac{\partial \alpha_{12}}{\partial p_2} = 0; & \dfrac{\partial \alpha_{34}}{\partial p_4} + \dfrac{\partial \alpha_{13}}{\partial p_1} = 0; & \dfrac{\partial \alpha_{13}}{\partial p_3} - \dfrac{\partial \alpha_{12}}{\partial p_2} = 0; \\ \dfrac{\partial \alpha_{41}}{\partial p_1} + \dfrac{\partial \alpha_{24}}{\partial p_2} = 0; & \dfrac{\partial \alpha_{24}}{\partial p_4} - \dfrac{\partial \alpha_{23}}{\partial p_3} = 0. & \end{array}\right.$$

Hence the α's are given by

$$(50)\quad \left\{\begin{array}{l} \alpha_{12} = p_2\phi_1 - p_1\phi_2;\ \alpha_{23} = p_3\phi_2 - p_2\phi_3;\ \alpha_{34} = p_4\phi_3 - p_3\phi_4; \\ \alpha_{41} = p_1\phi_4 - p_4\phi_1;\ \alpha_{13} = p_3\phi_1 - p_1\phi_3;\ \alpha_{24} = p_4\phi_2 - p_2\phi_4; \end{array}\right.$$

where the ϕ's are arbitrary functions of the coördinates x_1, x_2, x_3, x_4 only. Combining (50) and (46), we must have

$$(51)\qquad \frac{F_1 - \phi_1}{p_1} = \frac{F_2 - \phi_2}{p_2} = \frac{F_3 - \phi_3}{p_3} = \frac{F_4 - \phi_4}{p_4}$$

Letting K stand for these equal ratios, (51) may be written

$$(52)\qquad F_i = \phi_i + p_i K, \quad (i = 1, 2, 3, 4).$$

Multiplying by p_i and summing with respect to i, we get

$$\sum_i p_i F_i = \sum_i p_i \phi_i + K \sum_i p_i^2,$$

and remembering that

$$\sum_i (x'_i)^2 = 1, \quad \sum_i x'_i x''_i = 0$$

or

$$\sum_i p^2_i = 1, \quad \sum_i p_i F_i = 0,$$

we have

$$K = -\sum_i p_i \phi_i,$$

and (52) becomes

$$(53) \qquad F_i = \phi_i - p_i \sum_k p_k \phi_k, \quad (i = 1, 2, 3, 4).$$

We still have to satisfy equations (44). Substituting the values of the F's as given by (52) into (44), these reduce to

$$(54) \quad \begin{cases} p_4\left(\dfrac{\partial\phi_2}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_2}\right) + p_3\left(\dfrac{\partial\phi_4}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_4}\right) - p_2\left(\dfrac{\partial\phi_4}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_4}\right) = 0 \\ p_4\left(\dfrac{\partial\phi_3}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_3}\right) - p_3\left(\dfrac{\partial\phi_4}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_4}\right) + p_1\left(\dfrac{\partial\phi_4}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_4}\right) = 0 \\ p_4\left(\dfrac{\partial\phi_1}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_1}\right) + p_2\left(\dfrac{\partial\phi_4}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_4}\right) - p_1\left(\dfrac{\partial\phi_4}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_4}\right) = 0 \\ p_1\left(\dfrac{\partial\phi_2}{\partial x_3} - \dfrac{\partial\phi_3}{\partial x_2}\right) + p_2\left(\dfrac{\partial\phi_3}{\partial x_1} - \dfrac{\partial\phi_1}{\partial x_3}\right) + p_3\left(\dfrac{\partial\phi_1}{\partial x_2} - \dfrac{\partial\phi_2}{\partial x_1}\right) = 0 \end{cases}$$

Since these equations are to hold for arbitrary values of the p's, (the only condition being $p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1$), and since the expressions in parentheses are independent of the p's, we must have

$$(55) \quad \begin{cases} \dfrac{\partial\phi_1}{\partial x_2} = \dfrac{\partial\phi_2}{\partial x_1}; \quad \dfrac{\partial\phi_2}{\partial x_3} = \dfrac{\partial\phi_3}{\partial x_2}; \quad \dfrac{\partial\phi_3}{\partial x_4} = \dfrac{\partial\phi_4}{\partial x_3}; \\ \dfrac{\partial\phi_4}{\partial x_1} = \dfrac{\partial\phi_1}{\partial x_4}; \quad \dfrac{\partial\phi_1}{\partial x_3} = \dfrac{\partial\phi_3}{\partial x_1}; \quad \dfrac{\partial\phi_2}{\partial x_4} = \dfrac{\partial\phi_4}{\partial x_2}. \end{cases}$$

Hence, the ϕ's are partial derivatives of a single function $L(x_1, x_2, x_3, x_4)$, i. e.

$$(56) \qquad \phi_1 = \frac{\partial L}{\partial x_1}; \quad \phi_2 = \frac{\partial L}{\partial x_2}; \quad \phi_3 = \frac{\partial L}{\partial x_3}; \quad \phi_4 = \frac{\partial L}{\partial x_4}.$$

Introducing these values into (53), we finally get

$$(57) \qquad x_i'' = \frac{\partial L}{\partial x_i} - x_i' \sum_k \frac{\partial L}{\partial x_k} x_k', \quad (i = 1, 2, 3, 4),$$

and on comparison with (13), we note that these are the differential equations of our natural family in a euclidean space of four dimensions. We may therefore state the converse

THEOREM. *If a system of* ∞^6 *curves (one passing through each point in each direction) in a euclidean space of four dimensions is such that those* ∞^3 *curves of the system which meet an arbitrary hypersurface (space of three dimensions) orthogonally, always form a normal hypercongruence, the system is of the natural type.*

The Lipschitz theorem (§6) shows that the systems of the natural type actually have this property. In our discussion we have only considered the vanishing of α_1 in the condition (38) for a normal hypercongruence. We may therefore state a much stronger converse.

THEOREM. *If a system of* ∞^6 *curves in a euclidean space of four dimensions is such that those* ∞^3 *curves of the system which meet an arbitrary hypersurface orthogonally, always meet some infinitesimally neighboring hypersurface orthogonally, the system is of the natural type.*

As stated in the last theorem the weaker requirement of orthogonality must hold for *all* hypercongruences of the system, for a hypercongruence may meet two neighboring surfaces orthogonally (i. e. be approximately normal) without meeting ∞^1 hypersurfaces orthogonally.

§10. **Proof of the converse theorem in a euclidean space of n dimensions.** We return to our $(n-1)(n-2)/2$ conditions

$$(41)\quad \sum_{il}\frac{\partial F_i}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial f_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\sum_{il}\frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)=0,$$
$$(r,k=1,2,\ldots,n-1)$$

for a normal hypercongruence. We may evidently place each of the two summations in the left member of this equation equal to zero.

By grouping the terms in the first summation, this may be written

$$(58)\quad \sum_{il}\left(\frac{\partial F_i}{\partial x_l}-\frac{\partial F_l}{\partial x_i}\right)\left(\frac{\partial f_i}{\partial u_k}\frac{\partial f_l}{\partial u_r}-\frac{\partial f_i}{\partial u_r}\frac{\partial f_l}{\partial u_k}\right)=0, (r,k=1,2,\ldots,n-1).$$

To eliminate the f's we proceed as follows. Solve the $(n-1)$ equations

$$(33)\quad \sum_i P_i\frac{\partial f_i}{\partial u_r}=0,\quad (r=1,2,\ldots,n-1)$$

for the ratios of the P's. If we use the notation

$$(59)\qquad c_{\lambda\mu} = \frac{\partial f_\lambda}{\partial u_\mu}, \quad (c_{\lambda\mu} \pm c_{\mu\lambda})$$

equations (33) and (58) may be written

$$(60)\quad c_{1r}p_1 + c_{2r}p_2 + c_{3r}p_3 + \ldots\ldots + c_{nr}p_n = 0, (r = 1, 2, \ldots, n-1);$$

$$(61)\quad \sum_{il}\left(\frac{\partial F_i}{\partial x_l} - \frac{\partial F_l}{\partial x_i}\right)\begin{vmatrix} c_{ik} & c_{lk} \\ c_{ir} & c_{lr} \end{vmatrix} = 0, (r, k = 1, 2, \ldots\ldots, n-1).$$

From the $(n-1)$ homogeneous equations (60) we see that the p's are proportional to the determinants of the matrix

$$(62)\qquad \left\|\begin{matrix} c_{11} & \ldots c_{i1} & \ldots c_{l1} & \ldots c_{m1} & \ldots c_{n1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1k} & \ldots c_{ik} & \ldots c_{lk} & \ldots c_{mk} & \ldots c_{nk} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1r} & \ldots c_{ir} & \ldots c_{lr} & \ldots c_{mr} & \ldots c_{nr} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1,\,n-1} \ldots & c_{i,\,n-1} \ldots & c_{l,\,n-1} \ldots & c_{m,\,n-1} \ldots & c_{n,\,n-1} \end{matrix}\right\|$$

so that, e. g., if we omit the column headed by c_{m1} we get

$$(63)\quad p_m : (-1)^{m+1}\begin{vmatrix} c_{11} & \ldots c_{m-1,\,2} & c_{m+1,\,1} & \ldots c_{n1} \\ c_{12} & \ldots c_{m-1,\,2} & c_{m+1,\,2} & \ldots c_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_{1,\,n-1} \ldots & c_{m-1,\,n-1} & c_{m+1,\,n-1} \ldots & c_{n,\,n-1} \end{vmatrix}$$

Consider two arrays, the minors of the $2d$ order and the corresponding co-minors of the $(n-3)\,d$ order in q_m, where q_m is the determinant occurring in the above expression for p_m, so that

$$(64)\qquad p_m = (-1)^{m+1}\, q_m.$$

If we designate by $c_{ik,\,lr}$ the minor of $2d$ order,

$$(65)\qquad c_{ik,\,lr} = \begin{vmatrix} c_{ik} & c_{lk} \\ c_{ir} & c_{lr} \end{vmatrix}$$

and by $C_{ik,\,lr}$ its corresponding co-minor of the $(n-3)\,d$ order, and expand by Laplace's theorem[13] in terms of the $(n-1)\,(n-2)/2$ minors of the $2d$ order formed from the columns headed c_{i1} and c_{l1}, we get

$$(66)\quad q_m = (-1)^v\,[c_{i1,\,l2}\,C_{i1,\,l2} + c_{i1,\,l3}\,C_{i1,\,l3} + \ldots\ldots + c_{i\,n-2,\,l\,n-1}\,C_{i\,n-2,\,l\,n-1}],$$

where v is the sum of the columns in p_m headed by c_{i1} and c_{l1}.

From the matrix (62), we see that the determinants q_m, q_l, and q_i are formed by omitting the columns headed c_{m1}, c_{l1}, and c_{i1}, respectively, and that, therefore, the minors of the $2d$ order

$$c_{lk,\,mr} = \begin{vmatrix} c_{lk} & c_{mk} \\ c_{lr} & c_{mr} \end{vmatrix} \text{ and } c_{ik,\,mr} = \begin{vmatrix} c_{ik} & c_{mk} \\ c_{ir} & c_{mr} \end{vmatrix}$$

in the expansions for q_i and q_l, respectively, have the same co-minors of the $(n-3)\,d$ order that the minors of $2d$ order, $c_{ik,\,lr}$, have in the expansion for q_m. Designating these common co-minors by C_{12}, $C_{13}, \ldots\ldots, C_{n-2,\,n-1}$, we may now write the expansions

$$(67)\quad \begin{cases} q_m = (-1)^{i+l} & [c_{i1,\,l2}\,C_{12} + c_{i1,\,l3}\,C_{13} + \ldots\ldots + c_{i\,n-2,\,l\,n-1}\,C_{n-2,\,n-1}] \\ q_l = (-1)^{i+(m-1)} & [c_{il,\,m2}\,C_{12} + c_{il,\,m3}\,C_{13} + \ldots\ldots + c_{i\,n-2,\,m\,n-1}\,C_{n-2,\,n-1}] \\ q_i = (-1)^{(l-1)+(m-1)} & [c_{l1,\,m2}\,C_{12} + c_{l1,\,m3}\,C_{13} + \ldots\ldots + c_{l\,n-2,\,m\,n-1}\,C_{n-2,\,n-1}] \end{cases}$$

On the other hand, if the minors of the $2d$ order formed from two columns which are not headed by two of the three terms c_{m1}, c_{l1}, c_{i1}, are multiplied by the above co-minors of the $(n-3)d$ order, C_{12}, C_{13}, $\ldots\ldots$, $C_{n-2,\,n-1}$, and added, we shall have the expansion of a determinant with two columns alike, and hence the sum will be equal to zero.

Let us apply this discussion to the equations (61), involving the minors of the $2d$ order. If we expand these equations, and multiply the $(n-1)\,(n-2)/2$ equations by the above $(n-1)\,(n-2)/2$ co-minors of the $(n-3)\,d$ order, C_{12}, $C_{13}, \ldots\ldots, C_{n-2,\,n-1}$, in succession, and then add, we shall evidently get *three* terms only — all the sums will vanish except those giving q_m, q_l, and q_i. Equations (61) thus assume the form

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[13] See R. F. Scott, A treatise on the theory of determinants, 1880, chapt. III.

$$(68)\quad \left(\frac{\partial F_i}{\partial x_l}-\frac{\partial F_l}{\partial x_i}\right)(-1)^{i+l}q_m+\left(\frac{\partial F_i}{\partial x_m}-\frac{\partial F_m}{\partial x_i}\right)(-1)^{i+m-1}q_l+$$
$$\left(\frac{\partial F_l}{\partial x_m}-\frac{\partial F_m}{\partial x_l}\right)(-1)^{l+m-2}q_i=0$$

Substituting

$$q_m=(-1)^{m+1}p_m;\quad q_l=(-1)^{l+1}p_l;\quad q_i=(-1)^{i+1}p_i,$$

we get, finally,

$$(69)\quad \left(\frac{\partial F_i}{\partial x_l}-\frac{\partial F_l}{\partial x_i}\right)p_m-\left(\frac{\partial F_i}{\partial x_m}-\frac{\partial F_m}{\partial x_i}\right)p_l+\left(\frac{\partial F_l}{\partial x_m}-\frac{\partial F_m}{\partial x_l}\right)p_i=0$$

where i, l, m may take all values from 1 to n.

Let us now consider the vanishing of the 2*d* summation in (41), viz.,

$$(70)\quad \sum_{il}\frac{\partial F_i}{\partial p_l}\left(\frac{\partial P_l}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_l}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)=0,\ (r, k=1, 2, \ldots, n-1)$$

We must introduce here the identities

$$(33)\qquad \sum_i P_i\frac{\partial f_\iota}{\partial u_r}=0$$

$$(34)\qquad \sum_i\left(\frac{\partial P_i}{\partial u_k}\frac{\partial f_i}{\partial u_r}-\frac{\partial P_i}{\partial u_r}\frac{\partial f_i}{\partial u_k}\right)=0$$

Expanding (70) we get

$$(71)\quad \sum_i\frac{\partial F_i}{\partial p_1}\left(\frac{\partial P_1}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_1}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\sum_i\frac{\partial F_i}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_r}-\frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\ldots$$
$$+\sum_i\frac{\partial F_i}{\partial P_m}\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k}-\frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)+\ldots=0,\ (r, k=1, 2, \ldots, n-1),$$

and we may reduce the number of terms by eliminating the coefficients of $\frac{\partial F_n}{\partial p_1}, \frac{\partial F_1}{\partial p_2}, \frac{\partial F_2}{\partial p_3}, \ldots, \frac{\partial F_{m-1}}{\partial p_m}, \ldots,$ from the sums in (71). Thus to eliminate

$$\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_1}{\partial u_k}-\frac{\partial P_2}{\partial u_k}\frac{\partial f_1}{\partial u_r}\right),$$

the coefficient of $\frac{\partial F_1}{\partial p_2}$ in the second sum, we have from (33)

$$\frac{\partial f_1}{\partial u_k} = -\sum_{i=2}^{n} \frac{p_i}{p_1}\frac{\partial f_i}{\partial u_k}; \qquad \frac{\partial f_1}{\partial u_r} = -\sum_{i=2}^{n} \frac{p_i}{p_1}\frac{\partial f_i}{\partial u_r};$$

hence,

$$\frac{\partial F_i}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_1}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_1}{\partial u_r}\right) = -\sum_{i=2}^{n} \frac{p_i}{p_1}\frac{\partial F_1}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right),$$

and

$$\begin{aligned}\sum_i \frac{\partial F_i}{\partial p_2}\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) &= \sum_{i=2}^{n}\left[\frac{\partial F_i}{\partial p_2} - \frac{p_i}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)\\ &= \left[\frac{\partial F_2}{\partial p_2} - \frac{p_2}{p_1}\frac{\partial F_1}{\partial p_2}\right]\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_2}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_2}{\partial u_r}\right) + \sum_{i=3}^{n}\left[\frac{\partial F_i}{\partial p_2} - \frac{p_i}{p_1}\frac{\partial F_1}{\partial p_2}\right]\\ &\qquad\left(\frac{\partial P_2}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_2}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)\end{aligned}$$

Similarly, we may eliminate the coefficient of $\frac{\partial F_{m-1}}{\partial p_m}$ in the mth sum in (71), and thus find

$$(72)\quad \begin{aligned}\sum_i \frac{\partial F_i}{\partial p_m}\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right) &= \left[\frac{\partial F_m}{\partial p_m} - \frac{p_m}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_m}{\partial u_k}\right.\\ &\left.- \frac{\partial P_m}{\partial u_k}\frac{\partial f_m}{\partial u_r}\right) + \sum_{i=1\ldots m-1}^{m+1\ldots n}\left[\frac{\partial F_i}{\partial p_m} - \frac{p_i}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right).\end{aligned}$$

We may consider this as a typical term, letting m assume all values from 1 to n cyclically, or $m = 1 \to 2 \to \ldots . \to n$. Substituting in (71) and noting that the coefficients

$$\frac{\partial P_m}{\partial u_r}\frac{\partial f_m}{\partial u_k} - \frac{\partial P_m}{\partial u_k}\frac{\partial f_m}{\partial u_r}$$

are still related by the identity (34), whereby

$$\frac{\partial P_n}{\partial u_r}\frac{\partial f_n}{\partial u_k} - \frac{\partial P_n}{\partial u_k}\frac{\partial f_n}{\partial u_r} = -\sum_{i=1}^{n-1}\left(\frac{\partial P_i}{\partial u_r}\frac{\partial f_i}{\partial u_k} - \frac{\partial P_i}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)$$

equations (71) may finally be written

$$(72)\quad \sum_{m=1}^{n-1}\left[\left(\frac{\partial F_m}{\partial p_m}-\frac{p_m}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right)-\left(\frac{\partial F_n}{\partial p_n}-\frac{p_n}{p_{n-1}}\frac{\partial F_{n-1}}{\partial p_n}\right)\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_m}{\partial u_k}\right.$$
$$\left.-\frac{\partial P_m}{\partial u_k}\frac{\partial f_m}{\partial u_r}\right)+\sum_{m=1}^{n}\sum_{i=1\dots m-1}^{m+1\dots n}\left[\frac{\partial F_i}{\partial p_m}-\frac{p_i}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m}\right]\left(\frac{\partial P_m}{\partial u_r}\frac{\partial f_i}{\partial u_k}\right.$$
$$\left.-\frac{\partial P_m}{\partial u_k}\frac{\partial f_i}{\partial u_r}\right)=0.$$

In (72) all the coefficients in parentheses may now be taken as independent quantities, so that for (72) to vanish identically, we must have the vanishing of all the expressions in brackets. Hence

$$p_{m-1}\frac{\partial F_i}{\partial p_m}-p_i\frac{\partial F_{m-1}}{\partial p_m}=0,\quad \begin{pmatrix} i=1,\dots,m-1,\ m+1\dots n \\ m=1\rightarrow 2\rightarrow\dots\rightarrow n \end{pmatrix}.$$

By successive elimination of $\frac{\partial F_{m-1}}{\partial p_m}$, we may write these

$$p_k\frac{\partial F_i}{\partial p_m}-p_i\frac{\partial F_k}{\partial p_m}=0,\quad \begin{pmatrix} i,k=1,\dots,m-1,\ m+1\dots n \\ m=1,2,\dots,n \end{pmatrix}$$

or

$$\frac{\partial}{\partial p_m}(p_k F_i-p_i F_k)=0,\quad \begin{pmatrix} i,k=1,\dots,m-1,\ m+1\dots n \\ m=1,2,\dots,n \end{pmatrix}$$

so that $(p_k F_i - p_i F_k)$ can only contain p_i and p_k, and hence,

$$(73)\quad p_k F_i - p_i F_k = \alpha_{ik}(p_i, p_k, x_1, x_2, \dots, x_n),\quad (i,k=1,2,\dots,n)\ (i \pm k)$$

where

$$(74)\quad \alpha_{ik} = -\alpha_{ki}.$$

From (72) we further have

$$\frac{\partial F_n}{\partial p_n}-\frac{p_n}{p_{n-1}}\frac{\partial F_{n-1}}{\partial p_n}=\frac{\partial F_m}{\partial p_m}-\frac{p_m}{p_{m-1}}\frac{\partial F_{m-1}}{\partial p_m},\quad (m=1\rightarrow 2\rightarrow\dots\rightarrow n).$$

These are conditions on the forms of the α's in (73). By successive subtractions, these may be written

$$(75)\qquad \frac{\partial F_i}{\partial p_1} - \frac{p_i}{p_{i-1}}\frac{\partial F_{i-1}}{\partial p_i} = \frac{\partial F_k}{\partial p_k} - \frac{p_k}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_k} \quad (i, k = 1 \to 2 \to \ldots \to n).$$

If $k \pm i + 1$, we may replace $\frac{\partial F_{i-1}}{\partial p_i}$ by $\frac{p_{i-1}}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_i}$, and (75) becomes

$$\frac{\partial F_i}{\partial p_i} - \frac{p_i}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_i} = \frac{\partial F_k}{\partial p_k} - \frac{p_k}{p_{k-1}}\frac{\partial F_{k-1}}{\partial p_k}$$

and by subtraction of F_{k-1} from both members,

$$\frac{\partial}{\partial p_i}(p_{k-1}F_i - p_i F_{k-1}) = \frac{\partial}{\partial p_k}(p_{k-1}F_k - p_k F_{k-1}),$$

or

$$(76)\qquad \frac{\partial \alpha_{i,\,k-1}}{\partial p_i} = \frac{\partial \alpha_{k,\,k-1}}{\partial p_k}; \quad (k \pm i + 1).$$

If $k = i + 1$, we may replace $\frac{\partial F_{k-1}}{\partial p_k}$ by $\frac{p_{k-1}}{p_{i-1}}\frac{\partial F_{i-1}}{\partial p_k}$, and subtracting F_{i-1} from both members, (75) becomes

$$(77)\qquad \frac{\partial \alpha_{i,\,i-1}}{\partial p_i} = \frac{\partial \alpha_{k,\,i-1}}{\partial p_k}, \quad (k = i + 1).$$

From (76) and (77) and with the help of (74), we may now write the series of conditions on the α's:

$$(78)\qquad \frac{\partial \alpha_{12}}{\partial p_1} + \frac{\partial \alpha_{23}}{\partial p_3} = 0; \quad \frac{\partial \alpha_{23}}{\partial p_2} + \frac{\partial \alpha_{34}}{\partial p_4} = 0; \quad \ldots\ldots$$

$$\frac{\partial \alpha_{i+1,\,i}}{\partial p_{i+1}} + \frac{\partial \alpha_{ik}}{\partial p_k} = 0; \quad \frac{\partial \alpha_{ik}}{\partial p_i} + \frac{\partial \alpha_{k,\,k+1}}{\partial p_{k+1}} = 0; \quad \ldots\ldots$$

Hence the α's are given by

$$(79)\qquad \alpha_{ik} = p_k \phi_i - p_i \phi_k, \quad (i, k = 1, 2, \ldots, n)$$

where the ϕ's are arbitrary functions of the coördinates $x_1, x_2, \ldots, x_n$ only.

Combining (79) and (73), we must have

$$p_k F_i - p_i F_k = p_k \phi_i - p_i \phi_k \qquad (r, k = 1, 2, \ldots, n)$$

or

$$(80) \quad \frac{F_1 - \phi_1}{p_1} = \frac{F_2 - \phi_2}{p_2} = \ldots\ldots = \frac{F_i - \phi_i}{p_i} = \ldots\ldots = \frac{F_k - \phi_k}{p_k} = \ldots\ldots = \frac{F_n - \phi_n}{p_n}.$$

Letting K stand for these equal ratios, (80) may be written

$$(81) \qquad F_i = \phi_i + p_i K, \qquad (i = 1, 2, \ldots, n).$$

Multiplying by p_i and summing with respect to i, we get

$$\sum_i p_i F_i = \sum_i p_i \phi_i + K \sum_i p_i^2,$$

and remembering that

$$\sum_i p_i^2 = \sum_i (x_i')^2 = 1; \quad \sum_i p_i F_i = \sum_i x_i' x_i'' = 0,$$

we have

$$K = -\sum_i p_i \phi_i,$$

and (81) becomes

$$(82) \qquad F_i = \phi_i - p_i \sum_k p_k \phi_k \qquad (i = 1, 2, \ldots, n).$$

Now, we must still satisfy equations (69), and this will introduce conditions on the forms of the ϕ's. Substituting the values of the F's as given by (81) into (69), these reduce to

$$(83) \quad \left(\frac{\partial \phi_i}{\partial x_l} - \frac{\partial \phi_l}{\partial x_i}\right) p_m - \left(\frac{\partial \phi_i}{\partial x_m} - \frac{\partial \phi_m}{\partial x_i}\right) p_l + \left(\frac{\partial \phi_l}{\partial x_m} - \frac{\partial \phi_m}{\partial x_l}\right) p_i = 0,$$
$$(i, l, m = 1, 2, \ldots, n).$$

Since these equations are to hold for abitrary values of the p's (the only condition being $p_1^2 + p_2^2 + \ldots + p_n^2 = 1$), and since the expressions in parentheses are independent of the p's, we must have

$$(84) \qquad \frac{\partial \phi_i}{\partial x_l} = \frac{\partial \phi_l}{\partial x_i}, \quad (i, l = 1, 2, \ldots, n)$$

Hence, the ϕ's are partial derivatives of a single function $L(x_1,x_2,\ldots,x_n)$, or

$$\phi_i = \frac{\partial L}{\partial x_i}, \quad (i = 1, 2,\ldots, n). \tag{85}$$

Introducing these values into (82), we finally get

$$x_i'' = \frac{\partial L}{\partial x_i} - x_i' \sum_k \frac{\partial L}{\partial x_k} x_k' \qquad (i = 1, 2, \ldots, n), \tag{86}$$

and on comparison with (13), we note that these are the differential equations of our natural family in a euclidean space of n dimensions. We may therefore state the converse

THEOREM. *If a system of* $\infty^{2(n-1)}$ *curves (one passing through each point in each direction) in a euclidean space of n dimensions is such that those* ∞^{n-1} *curves of the system which meet an arbitrary hypersurface (space of* $n - 1$ *dimensions) orthogonally, always form a normal hypercongruence, the system is of the natural type.*

The Lipschitz theorem (§6) shows that the systems of the natural type actually have this property. Since we have only considered the vanishing of α_1 in the condition (38), we may here, analogous to the case of four dimensions, state a much stronger converse theorem.

§11. **Proof of the converse theorem in any space.** In §7 we derived the necessary analytic conditions

$$T\left(\frac{\partial U_k}{\partial u_r} - \frac{\partial U_r}{\partial u_k}\right) + U_k\left(\frac{\partial U_r}{\partial t} - \frac{\partial T}{\partial u_r}\right) + U_r\left(\frac{\partial T}{\partial u_k} - \frac{\partial U_k}{\partial t}\right) = 0, \tag{26'}$$

$$(r, k = 1, 2, \ldots, n - 1)$$

where

$$T = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial t}; \quad U_r = \sum_{\lambda\mu} a_{\lambda\mu} \frac{\partial X_\lambda}{\partial t}\frac{\partial X_\mu}{\partial t} = 0, \tag{25''}$$

$$(r = 1, 2,\ldots, n - 1),$$

for a hypercongruence of curves

$$X_i = X_i(u_1, u_2, \ldots, u_{n-1}, t), \qquad (i = 1, 2, \ldots, n) \tag{20'}$$

to be of the normal type. We may proceed exactly as in §8 and apply these conditions to a family of ∞^{n-1} curves picked out from the system of $\infty^{2(n-1)}$ curves defined by

$$(27)\quad x_i'' = F_i\,(x_1, x_2, \ldots, x_n;\ x_1', x_2', \ldots, x_n'),\quad (i = 1, 2, \ldots, n),$$

and passing out orthogonally to an arbitrary hypersurface

$$(32)\qquad x_i = f_i\,(u_1, u_2, \ldots, u_{n-1}),\quad (i = 1, 2, \ldots, n).$$

This hypercongruence may be written

$$(36)\quad X_i = f_i + P_i\,t + \tfrac{1}{2}\,\overline{F}_i t^2 + \tfrac{1}{6}\,\overline{M}_i t^3 + \ldots,\qquad (i = 1, 2, \ldots, n),$$

where

$$(37)\quad \overline{F}_i\,(u_1, u_2, \ldots, u_{n-1}) = F_i\,(f_1, f_2, \ldots, f_n;\ P_1, P_2 \ldots, P_n)$$

and the P's are connected by identities similar to (33) and (34) but of a more general form, viz.,

$$(33')\quad \sum_{\lambda\mu} a_{\lambda\mu}\,P_\lambda\,P_\mu = 1;\quad \sum_{\lambda\mu} a_{\lambda\mu}\,P_\lambda\,\frac{\partial f_\mu}{\partial u_r} = 0,\ (r = 1, 2, \ldots, n-1);$$

$$(34')\quad \sum_{\lambda\mu} P_\mu\left(2\,a_{\lambda\mu}\frac{\partial P_\lambda}{\partial u_r} + P_\lambda\frac{\partial a_{\lambda\mu}}{\partial u_r}\right) = 0,\quad (r = 1, 2, \ldots n-1);$$

$$\sum_{\lambda\mu}\frac{\partial f_\mu}{\partial u_\mu}\left(a_{\lambda\mu}\frac{\partial P_\lambda}{\partial u_k} + P_\lambda\frac{\partial a_{\lambda\mu}}{\partial u_k}\right) = \sum_{\lambda\mu}\frac{\partial f_\mu}{\partial u_k}\left(a_{\lambda\mu}\frac{\partial P_\lambda}{\partial u_r} + P_\lambda\,\frac{\partial a_{\lambda\mu}}{\partial u_r}\right),$$
$$(r, k = 1,\ 2, \ldots,\ n-1).$$

Before applying the conditions of orthogonality to the curves (36) we shall introduce considerable simplification in the work by introducing a set of new quantities, Q_i, defined by the linear transformation

$$(87)\qquad Q_\mu = \sum_\lambda a_{\lambda\mu}\,P_\lambda,\quad \text{and hence, } P_\mu = \sum_\lambda A_{\lambda\mu}\,Q_\lambda$$

where $A_{\lambda\mu}$ is the minor of $a_{\lambda\mu}$ in the determinant $a = |a_{\lambda\mu}|$ divided by a itself, so that

$$(88)\qquad \sum_\mu a_{i\mu}\,A_{\lambda\mu} = 0,\quad \text{if } i \pm \lambda\ ;\quad \sum_\mu a_{\lambda\mu}\,A_{\lambda\mu} = 1.$$

Now, the identities (33′) and (34′) take the much simpler form

$$(89)\quad \sum_\mu Q_\mu\,P_\mu = 1;\quad \sum_\mu Q_\mu\,\frac{\partial f_\mu}{\partial u_r} = 0,\quad (r = 1, 2, \ldots, n-1);$$

$$(90) \qquad \sum_{\mu}\left(Q_{\mu}\frac{\partial P_{\mu}}{\partial u_r}+P_{\mu}\frac{\partial Q_{\mu}}{\partial u_r}\right)=0, \quad (r=1,2,\ldots,n-1);$$

$$\sum_{\mu}\left(\frac{\partial Q_{\mu}}{\partial u_k}\frac{\partial f_{\mu}}{\partial u_r}-\frac{\partial Q_{\mu}}{\partial u_r}\frac{\partial f_{\mu}}{\partial u_k}\right)=0, \quad (r,k=1,2,\ldots,n-1).$$

Applying our condition of orthogonality, we get

$$\frac{\partial X_{\lambda}}{\partial t}=P_{\lambda}+t\overline{F}_{\lambda}+\ldots; \qquad \frac{\partial X_{\mu}}{\partial t}=P_{\mu}+t\overline{F}_{\mu}+\ldots.;$$

$$\frac{\partial X_{\mu}}{\partial u_r}=\frac{\partial f_{\mu}}{\partial u_r}+t\frac{\partial P_{\mu}}{\partial u_r}+\ldots; \qquad \frac{\partial X_{\mu}}{\partial u_k}=\frac{\partial f_{\mu}}{\partial u_k}+t\frac{\partial P_{\mu}}{\partial u_k}+\ldots;$$

$$T=\sum_{\lambda\mu}a_{\lambda\mu}P_{\lambda}P_{\mu}+t\sum_{\lambda\mu}a_{\lambda\mu}(P_{\lambda}\overline{F}_{\mu}+P_{\mu}\overline{F}_{\lambda})+\ldots=1+2t\sum_{\mu}Q_{\mu}\overline{F}_{\mu} +\ldots.;$$

$$U_r=\sum_{\lambda\mu}a_{\lambda\mu}P_{\lambda}\frac{\partial f_{\mu}}{\partial u_r}+t\sum_{\lambda\mu}a_{\lambda\mu}\left(P_{\lambda}\frac{\partial P_{\mu}}{\partial u_r}+\overline{F}_{\lambda}\frac{\partial f_{\mu}}{\partial u_r}\right)+\ldots.;$$

$$U_r=t\left(\sum_{\mu}Q_{\mu}\frac{\partial P_{\mu}}{\partial u_r}+\sum_{\lambda\mu}a_{\lambda\mu}\overline{F}_{\lambda}\frac{\partial f_{\mu}}{\partial u_r}\right)+\ldots.;$$

$$U_k=t\left(\sum_{\mu}Q_{\mu}\frac{\partial P_{\mu}}{\partial u_r}+\sum_{\lambda\mu}a_{\lambda\mu}\overline{F}_{\lambda}\frac{\partial f_{\mu}}{\partial u_k}\right)+\ldots.;$$

$$\frac{\partial U_r}{\partial u_k}=t\left[\sum_{\mu}\left(Q_{\mu}\frac{\partial^2 P_{\mu}}{\partial u_k\,\partial u_k}+\frac{\partial Q_{\mu}}{\partial u_k}\frac{\partial P_{\mu}}{\partial u_r}\right)+\sum_{\lambda\mu}a_{\lambda\mu}\overline{F}_{\lambda}\frac{\partial^2 f_{\mu}}{\partial u_k\,\partial u_r} +\sum_{\lambda\mu}\frac{\partial(a_{\lambda\mu}\overline{F}_{\lambda})}{\partial u_k}\frac{\partial f_{\mu}}{\partial u_r}\right]+\ldots.;$$

$$\frac{\partial U_k}{\partial u_r}=t\left[\sum_{\mu}\left(Q_{\mu}\frac{\partial^2 P_{\mu}}{\partial u_r\,\partial u_k}+\frac{\partial Q_{\mu}}{\partial u_r}\frac{\partial P_{\mu}}{\partial u_k}\right)+\sum_{\lambda\mu}a_{\lambda\mu}\overline{F}_{\lambda}\frac{\partial^2 f_{\mu}}{\partial u_r\,\partial u_k} +\sum_{\lambda\mu}\frac{\partial(a_{\lambda\mu}\overline{F}_{\lambda})}{\partial u_r}\frac{\partial f_{\mu}}{\partial u_k}\right]+\ldots..$$

The vanishing of the coefficient of the first power of t in the condition of orthogonality reduces to

$$(91) \sum_{\lambda\mu}\left[\frac{\partial(a_{\lambda\mu}\overline{F}_{\lambda})}{\partial u_r}\frac{\partial f_{\mu}}{\partial u_k}-\frac{\partial(a_{\lambda\mu}\overline{F}_{\lambda})}{\partial u_r}\frac{\partial f_{\mu}}{\partial u_k}\right]+\sum_{\mu}\left[\frac{\partial Q_{\mu}}{\partial u_r}\frac{\partial P_{\mu}}{\partial u_k}-\frac{\partial Q_{\mu}}{\partial u_k}\frac{\partial P_{\mu}}{\partial u_r}\right]=0, \quad (r,k=1,2,\ldots,n-1).$$

Now (37) may be written

$$(92)\quad \overline{F}_i(u_1, u_2, \ldots, u_{n-1}) = F_i\,(f_1, f_2, \ldots, f_n;\; P_1, P_2, \ldots, P_n) = F'_i\,(f_1, f_2, \ldots, f_n;\, Q_1, Q_2, \ldots, Q_n),$$

where F'_i represents the form which the function F_i takes when the P's are expressed in terms of the Q's, by (87). Then

$$(93)\qquad \begin{aligned} \frac{\partial \overline{F}_\lambda}{\partial u_r} &= \sum_l \frac{\partial F'_\lambda}{\partial x_l}\frac{\partial f_l}{\partial u_r} + \sum_l \frac{\partial F'_\lambda}{\partial q_l}\frac{\partial Q_l}{\partial u_r} \\ \frac{\partial \overline{F}_\lambda}{\partial u_k} &= \sum_l \frac{\partial F'_\lambda}{\partial x_l}\frac{\partial f_l}{\partial u_k} + \sum_l \frac{\partial F'_\lambda}{\partial q_l}\frac{\partial Q_l}{\partial u_k} \end{aligned}$$

with similar expressions for the partial derivatives of $a_{\lambda\mu}$, and (91) then becomes

$$(94)\quad \sum_{\lambda\mu l}\left[\frac{\partial(a_{\lambda\mu}F'_\lambda)}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k} - \frac{\partial f_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right) + \frac{\partial(a_{\lambda\mu}F'_\lambda)}{\partial q_l}\left(\frac{\partial Q_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k} - \frac{\partial Q_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right)\right] + \sum_\mu\left[\frac{\partial Q_\mu}{\partial u_r}\frac{\partial P_\mu}{\partial u_k} - \frac{\partial Q_\mu}{\partial u_k}\frac{\partial P_\mu}{\partial u_r}\right] = 0,$$

$$(r, k = 1, 2 \ldots, n-1).$$

These $(n-1)(n-2)/2$ conditions are therefore necessary conditions for a normal hypercongruence. Our problem is to find the forms of the functions F'_i or F_i in order that this condition of orthogonality should hold for *every* hypersurface taken as a base. We may simplify our problem by the following considerations. Since the geodesics in our space form a normal hypercongruence for every hypersurface as base, the equations of the geodesics

$$(18)\qquad \frac{d^2x_\lambda}{ds^2} = -\sum_{ik}\left\{ {ik \atop \lambda} \right\}\frac{dx_i}{ds}\frac{dx_k}{ds}, \quad (\lambda = 1, 2, \ldots, n)$$

or

$$(95)\quad \frac{d^2x_\lambda}{ds^2} = G_\lambda\,(f_1, f_2, \ldots, f_n;\; P_1, P_2, \ldots, P_n) = G'_\lambda\,(f_1, f_2, \ldots f_n;\; Q_1, Q_2, \ldots Q_n)$$

must satisfy conditions (94). Substituting G'_λ for F'_λ in (94), subtracting the result from (94) and writing

$$H'_\lambda = F'_\lambda - G'_\lambda \tag{96}$$

we get

$$\sum_{\lambda\mu l}\left[\frac{\partial(a_{\lambda\mu}H'_\lambda)}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}-\frac{\partial f_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right)+\frac{\partial(a_{\lambda\mu}H'_\lambda)}{\partial q_l}\left(\frac{\partial Q_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}-\frac{\partial Q_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right)\right]=0,\quad (r,k=1,2,\ldots,n-1). \tag{97}$$

Making the further linear transformation

$$J'_\mu = \sum_\lambda a_{\lambda\mu}H'_\lambda \tag{98}$$

equations (97) take the simple form

$$\sum_{l\mu}\frac{\partial J'_\mu}{\partial x_l}\left(\frac{\partial f_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}-\frac{\partial f_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right)+\sum_{l\mu}\frac{\partial J'_\mu}{\partial q_l}\left(\frac{\partial Q_l}{\partial u_r}\frac{\partial f_\mu}{\partial u_k}-\frac{\partial Q_l}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}\right)=0,\quad (r,k=1,2,\ldots,n-1). \tag{99}$$

Here the form of J' is to be determined if the f's are arbitrary functions, the only relations between the f's and the Q's being the identities

$$\sum_\mu Q_\mu\frac{\partial f_\mu}{\partial u_r}=0,\quad (r=1,2,\ldots,n-1); \tag{89}$$

$$\sum_\mu\left(\frac{\partial Q_\mu}{\partial u_k}\frac{\partial f_\mu}{\partial u_r}-\frac{\partial Q_\mu}{\partial u_r}\frac{\partial f_\mu}{\partial u_r}\right)=0,\quad (r,k=1,2,\ldots,n-1). \tag{90}$$

We note that equations (99) and their accompanying identities (89) and (90) have the same form as equations (41) and their accompanying identities (33) and (34), with i replaced by μ, F by J', and P by Q. The form of F is determined by (80) and (85), and we may therefore immediately write down the form of J', as

$$\frac{J'_1-\phi_1}{q_1}=\frac{J'_2-\phi_2}{q_2}=\ldots\ldots=\frac{J'_i-\phi_i}{q_i}=\ldots\ldots=\frac{J'_n-\phi_n}{q_n} \tag{100}$$

where the ϕ's are partial derivatives of a single function $L(x_1, x_2, \ldots x_n)$, or

$$\phi_i=\frac{\partial L}{\partial x_i},\quad (i=1,2,\ldots,n). \tag{101}$$

Letting K stand for the equal ratios in (100), we may write these

$$J'_i = \phi_i + q_i K, \quad (i = 1, 2, \ldots, n)$$

or, by (98)

$$\sum_{\lambda} a_{\lambda i} H'_{\lambda} = \phi_i + q_i K, \quad (i = 1, 2, \ldots n).$$

To solve these for H'_{λ}, we multiply by A_{li} and sum with respect to i, then

$$H'_l = \sum_i A_{li} \phi_i + K \sum_i A_{li} q_i.$$

Hence by (87)

$$H_l = \sum_i A_{li} \phi_i + K p_l.$$

To find the value of K, we now multiply by $a_{\lambda l} p_{\lambda}$, sum with respect to λ and l, and use the relation (88),

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} H_l = \sum_i p_i \phi_i + K \sum_{\lambda l} a_{\lambda l} p_{\lambda} p_l.$$

Now since

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} p_l = 1,$$

then, by total differentiation with respect to s, we may show that

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} F_l + \tfrac{1}{2} \sum_{i \lambda l} \frac{\partial a_{\lambda l}}{\partial x_i} p_i p_{\lambda} p_l = 0$$

or

$$\sum_{\lambda l} a_{\lambda l} p_{\lambda} F_l - \sum_{\lambda l} a_{\lambda l} p_{\lambda} G_l = 0 = \sum_{\lambda l} a_{\lambda l} p_{\lambda} H_l.$$

Hence,

$$K = -\sum_i p_i \phi_i$$

and

$$H_l = \sum_i A_{li} \phi_i - p_l \sum_i p_i \phi_i.$$

Finally, replacing H_l by $F_l - G_l$, G_l by its value from (18), and ϕ_i by its value from (101), we get

$$(102) \quad x''_l + \sum_{ik} \left\{ {ik \atop l} \right\} x'_i x'_k = \sum_i \frac{\partial L}{\partial x_i} (A_{li} - x'_l x'_i), \quad (l = 1, 2, \ldots, n)$$

and on comparison with (15) we note that these are the differential equations of our natural family in any space of n dimensions. We may therefore state the converse

THEOREM. *If a system of* $\infty^{2(n-1)}$ *curves (one passing through each point in each direction) in any curved space of* n *dimensions is such that those* ∞^{n-1} *curves of the system which meet on arbitrary hypersurface (space of* n − 1 *dimensions) orthogonally, always form a normal hypercongruence, the system is of the natural type.*

The Lipschitz theorem (§6) shows that the systems of the natural type actually have this property. Since we have only considered the vanishing of α_1 in the condition (38), we may here, analogous to the case in a euclidean space, state a much stronger converse theorem.

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Cambridge, Mass., January 1920.